Lecture 13:

Exit distributions

Ex 1. (Two-year college). At a local two-year college, 60% of first year students become sophomores, 25% remain in the first year, and 15% dropout; 70% of sophomores graduate and transfer to a four-year university, 20% remain sophomores and 10% dropout.

Q: What fraction of new students eventually graduate?

A: First, we write down the transition matrix:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ G \\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & G & D \end{array} \\ \begin{bmatrix} 0.25 & 0.6 & 0 & 0.15 \\ 0 & 0.2 & 0.7 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Let $h(x)$ be the probability that a student currently in state x eventually graduates. Then,

$$\begin{cases} h(1) = 0.25 \cdot h(1) + 0.6 \cdot h(2) ; \\ h(2) = 0.2\, h(2) + 0.7 \quad . \end{cases}$$

Solving the system of equations gives

$$\begin{cases} h(1) = 0.7 \\ h(2) = 0.875. \end{cases}$$

Therefore, there are 70% of new students eventually graduate.

Ex 2 (Tennis). In tennis the winner of a game is the first player to win 4 points, unless the score is 4:3, in which case the game must continue until one player is ahead by two points and wins the game. Suppose that a player wins points with probability 0.6 and successive points are independent.

Q: What's the probability the player will win the game if the score is tied 3:3? if she is ahead by one point? behind by one point?

A: We formula the state space of this Markov chain by the difference of the scores.

$$P = \begin{array}{c|ccccc} & -2 & -1 & 0 & 1 & 2 \\ \hline -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 & 0 \\ 1 & 0 & 0 & 0.4 & 0 & 0.6 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}$$

If we let $h(x)$ be the probability of the player winning when the difference of the score is x.

Then $h(2) = 1$, $h(-2) = 0$, and

$$h(x) = \mathbb{P}(\tau_2 < \tau_{-2} \mid X_0 = x)$$

$$= \sum_{y \in \mathcal{X}} \mathbb{P}(\tau_2 < \tau_{-2}, X_1 = y \mid X_0 = x)$$

why?

$$= \sum_{y \in \mathcal{X}} \mathbb{P}(\tau_2 < \tau_{-2} \mid X_1 = y, X_0 = x) \cdot P_{xy}$$

why?

$$= \sum_{y \in \mathcal{X}} \mathbb{P}(\tau_2 < \tau_{-2} \mid X_1 = y) \cdot P_{xy}$$

$$= \sum_{\substack{y \in \mathcal{X} \\ y \neq \{-2, 2\}}} h(y) \cdot P_{xy} + P_{x2} = \sum_{y} P_{xy} \cdot h(y).$$

Thus, $h(x)$ satisfies

$$\begin{cases} h(x) = \sum_y P_{xy} \cdot h(y), & \forall\, x = -1, 0, 1 ; \\ h(2) = 1 ; \\ h(-2) = 0 . \end{cases}$$

Plugging P_{xy} and $h(2)$, $h(-2)$, one has

$$(*) \quad \begin{cases} h(-1) = 0.6\, h(0) ; \\ h(0) = 0.6\, h(1) + 0.4\, h(-1) ; \\ h(1) = 0.6 + 0.4\, h(0) . \end{cases}$$

Solve the linear system and one has

$$h(1) = 0.8769, \quad h(0) = 0.6923, \quad h(-1) = 0.4154 .$$

Remark 13.1. The equations $(*)$ can be written as

$$\begin{cases} h(-1) - 0.6\, h(0) + 0 \cdot h(1) = 0 ; \\ -0.4\, h(-1) + h(0) - 0.6\, h(1) = 0 ; \\ 0 \cdot h(-1) - 0.4\, h(0) + h(1) = 0.6 . \end{cases}$$

This implies

$$\begin{bmatrix} h(-1) \\ h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} 1 & -0.6 & 0 \\ -0.4 & 1 & -0.6 \\ 0 & -0.4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0.6 \end{bmatrix}.$$

Let $C = \{-1, 0, 1\}$ be the set of non-absorbing states, and let Q be the restriction of P on C. Then the last equation is

$$\begin{bmatrix} h(-1) \\ h(0) \\ h(1) \end{bmatrix} = (I - Q)^{-1} \begin{bmatrix} 0 \\ 0 \\ 0.6 \end{bmatrix}.$$

Remark 13.2. Another method of solving $h(0)$ is, considering in two steps,

$$h(0) = 0.6^2 + 2(1-0.6)\cdot 0.6 \cdot h(0) + (1-0.6)^2 \cdot 0.$$

This implies $h(0) = \frac{9}{13}$.

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Remark 13.3. For the above two systems of equations in Ex 1 & Ex 2, we solved with unique solutions, which give the correct answer. The follow theorem insures this uniqueness, which is nice to know when we sometimes want to guess and then verify the answers.

In the following, we use $V_F := \min\{n > 0 : X_n \in F\}$ to denote the first time visiting the set $F \subseteq \mathcal{X}$, and $V_A \wedge V_B := \min\{V_A, V_B\}$.

Theorem 13.1. Consider a Markov chain with a state space $\mathcal{X}$. Let A and B be subsets of $\mathcal{X}$ and $C = \mathcal{X} \setminus (A \cup B)$ is finite. Suppose $h(a) = 1 \ \forall a \in A$, $h(b) = 0 \ \forall b \in B$, and $h(x) = \sum_{y \in \mathcal{X}} P_{xy} \cdot h(y) \quad \forall x \in C$.

If $\mathbb{P}_x(V_A \wedge V_B < \infty) > 0 \quad \forall x \in C$, then

$$h(x) = \mathbb{P}_x(V_A < V_B).$$

Proof. Let $T = V_A \wedge V_B$.

Claim 1: $\forall x \in C$, $\mathbb{P}_x(T<\infty)=1$.

Pf of Claim 1: Since $\{T<k\} \subseteq \{T<k+1\}$ $\forall k \geq 1$, and $\{T<\infty\} = \bigcup_{k=1}^{\infty} \{T<k\}$. By the continuity of $\mathbb{P}$, one has

$$\mathbb{P}_x(T<\infty) = \lim_{k\to\infty} \mathbb{P}_x(T<k).$$

For any fixed $x \in C$, since $\mathbb{P}_x(T<\infty)>0$, there exists $K_x \in \mathbb{N}$ such that

$$\mathbb{P}_x(T<k)>0 \qquad \forall k \geq K_x.$$

In particular, $\mathbb{P}_x(T<K_x)>0$ $\forall x \in C$.

Since C is finite, $K := \max_{x\in C}\{K_x\} < \infty$.

Thus, $\mathbb{P}_x(T<K) \geq \mathbb{P}_x(T<K_x) > 0$, $\forall x \in C$.

Again since C is finite, $\alpha := \min_{x\in C} \mathbb{P}_x(T<K) > 0$.

Therefore, $\mathbb{P}_x(T \leq K) \geq \mathbb{P}_x(T<K) \geq \alpha > 0$ $\forall x \in C$.

Recall

Suppose $\exists\, k > 1, \alpha > 0$, s.t
$\mathbb{P}_x(\tau_y \le k) \ge \alpha$
for all $x \in \mathcal{X}$. Then
$\mathbb{P}_x(\tau_y > nk) \le (1-\alpha)^n$,
$\forall n \ge 1$.

Thus, $\mathbb{P}_x(T > nK) \le (1-\alpha)^n$. $\forall x \in C$.

And $$\mathbb{P}_x(T < \infty) = 1 - \mathbb{P}_x(T = \infty)$$
$$= 1 - \lim_{n\to\infty} \mathbb{P}_x(T > nK)$$
$$= 1 - 0$$
$$= 1 \qquad \forall x \in C.$$

Claim 2: $h(x) = \mathbb{E}_x h(X_T)$.

Pf of Claim 2:

$$\mathbb{E}_x h(X_T) - \mathbb{E}_x h(X_0)$$
$$= \mathbb{E}_x \left\{ \sum_{n=0}^{\infty} [h(X_{n+1}) - h(X_n)] \cdot \mathbb{1}_{\{n<T\}} \right\}$$
$$\overset{(*)}{=} \sum_{n=0}^{\infty} \mathbb{E}_x \left\{ [h(X_{n+1}) - h(X_n)] \cdot \mathbb{1}_{\{n<T\}} \right\}$$

(*) By Fubini's Theorem

For each $n \in \mathbb{N}$, let $\mathcal{G}_n = (X_0 = x, X_1, \cdots, X_n)$, then the Tower Rule implies

$$\mathbb{E}_x \left\{ [h(X_{n+1}) - h(X_n)] \cdot \mathbb{1}_{\{n<T\}} \right\}$$
$$= \mathbb{E}_x \left[\mathbb{E}\left\{ [h(X_{n+1}) - h(X_n)] \cdot \mathbb{1}_{\{n<T\}} \mid \mathcal{G}_n \right\} \right]$$

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$$= \mathbb{E}_x\left[\mathbb{E}\left[h(X_{n+1}) - h(X_n) \mid \mathcal{G}_n \right] \cdot \mathbb{E}\left[\mathbb{1}_{\{n<T\}} \mid \mathcal{G}_n \right] \right]$$

$$= \mathbb{E}_x\left[\left\{ \mathbb{E}\left[h(X_{n+1}) \mid \mathcal{G}_n \right] - \mathbb{E}\left[h(X_n) \mid \mathcal{G}_n \right] \right\} \cdot \mathbb{E}\left[\mathbb{1}_{\{n<T\}} \mid \mathcal{G}_n \right] \right]$$

$$= \mathbb{E}_x\left[\left\{ \mathbb{E}\left[h(X_{n+1}) \mid X_n \right] - \mathbb{E}\left[h(X_n) \mid X_n \right] \right\} \cdot \mathbb{E}\left[\mathbb{1}_{\{n<T\}} \mid \mathcal{G}_n \right] \right]$$

$$= \mathbb{E}_x\left[\left\{ \sum_{y\in\mathcal{X}} h(y) \cdot \mathbb{P}(X_{n+1} = y \mid X_n) - h(X_n) \right\} \cdot \mathbb{E}\left[\mathbb{1}_{\{n<T\}} \mid \mathcal{G}_n \right] \right]$$

$$= \mathbb{E}_x\left[\left\{ \sum_{y\in\mathcal{X}} h(y) \cdot P_{X_n, y} - h(X_n) \right\} \cdot \mathbb{E}\left[\mathbb{1}_{\{n<T\}} \mid \mathcal{G}_n \right] \right]$$

$$= \mathbb{E}_x\left[\left\{ h(X_n) - h(X_n) \right\} \cdot \mathbb{E}\left[\mathbb{1}_{\{n<T\}} \mid \mathcal{G}_n \right] \right]$$

$$= 0.$$

Therefore, $\mathbb{E}_x h(X_T) - \mathbb{E}_x h(X_0) = \sum_{n=0}^{\infty} 0 = 0.$

Thus,
$$\begin{aligned} h(x) &= \mathbb{E}_x h(X_0) \\ &= \mathbb{E}_x h(X_T) \\ &= \sum_{a\in A} h(a)\, \mathbb{P}_x(X_T = a) + \sum_{b\in B} h(b) \cdot \mathbb{P}_x(X_T = b) \\ &= \sum_{a\in A} \mathbb{P}_x(X_T = a) \\ &= \mathbb{P}_x(X_T \in A) \\ &= \mathbb{P}_x(V_A < V_B). \quad \blacksquare \end{aligned}$$

Ex3. (Gambler's Ruin, revisit)

Consider a gambling game in which on any turn you win $\$1$ with probability $p \neq \frac{1}{2}$ or lose $\$1$ with probability $1-p$. Suppose further that you will quit playing if your fortune reaches $\$N$. Of course, the casino makes you stop when your fortune reaches $\$0$.

Q: What is $\mathbb{P}_x(V_N < V_0)$? i.e. the probability of exiting the game with fortune of $\$N$.

A: Let $h(x) = \mathbb{P}_x(V_N < V_0)$. Then we have

$$\begin{cases} h(x) = (1-p)\cdot h(x-1) + p\cdot h(x+1), & \forall 0<x<N; \\ h(0) = 0; \\ h(N) = 1. \end{cases}$$

To solve this, notice that

$$p\cdot\big(h(x+1) - h(x)\big) = (1-p)\big(h(x) - h(x-1)\big).$$

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Therefore, $h(x+1) - h(x) = \frac{1-p}{p}\left(h(x) - h(x-1)\right) \quad \forall\ 0 < x < N.$

Let $c = h(1) - h(0)$, and $\theta = \frac{1-p}{p}$, then

$$h(x+1) - h(x) = c\,\theta^x \qquad \forall\ 0 \le x < N.$$

Summing from $x = 0$ to $N-1$, one has

$$1 = h(N) - h(0) = \sum_{x=0}^{N-1} \left(h(x+1) - h(x) \right)$$

$$= \sum_{x=0}^{N-1} c\,\theta^x$$

$$= c \cdot \frac{1-\theta^N}{1-\theta}.$$

This implies $\quad c = \frac{1-\theta}{1-\theta^N}.$

For any $0 < x < N$, using the same trick,

$$h(x) = h(x) - h(0) = \sum_{i=0}^{x-1} h(i+1) - h(i)$$

$$= \sum_{i=0}^{x-1} c \cdot \theta^i = \frac{1-\theta^x}{1-\theta^N}.$$

Check that this formula also works for $x = 0, N$.

Thus, $\mathbb{P}_x(V_N < V_0) = \frac{1-\theta^x}{1-\theta^N}$, where $\theta = \frac{1-p}{p}$.

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Ex4. (Roulette) If we bet $\$1$ on red on a roulette wheel with 18 red, 18 black, and 2 green (i.e. 0 and 00) holes, we win $\$1$ with probability $\frac{18}{38} = \frac{9}{19}$ and lose $\$1$ with probability $\frac{10}{19}$. Suppose we bring $\$50$ to the casino wishing to reach $\$100$ before going bankrupt.

Q: What's the probability of success?

A: In this case, $\theta = \frac{1-p}{p} = \frac{10}{9}$.

Thus, $\mathbb{P}_{50}(V_{100} < V_0) = \dfrac{1-\left(\frac{10}{9}\right)^{50}}{1-\left(\frac{10}{9}\right)^{100}} = \dfrac{1}{1+\left(\frac{10}{9}\right)^{50}}$.

$$= \frac{1}{1+194.03} = \frac{1}{195.03} = 0.00513.$$

Q: Suppose the casino starts with a capital $\$100$. Then what's the probability of reaching $\$N$ before bankrupt?

A: $\mathbb{P}_{100}(V_N < V_0) = \dfrac{1-\left(\frac{9}{10}\right)^{100}}{1-\left(\frac{9}{10}\right)^{N}}$.

Remark 13.4. As $N \longrightarrow \infty$, $\mathbb{P}_{100}(V_N < V_0) \longrightarrow 1 - 0.9^{100}$.

Thus, $\mathbb{P}_{100}(V_0 < \infty) = 0.9^{100} = 2.66 \times 10^{-5}$.

If the casino starts with \$200, then

$$\mathbb{P}_{200}(V_0 < \infty) = (\mathbb{P}_{100}(V_0 < \infty))^2 = 7.055 \times 10^{-10}.$$

Lesson: Don't do gambling unless you have $p > \frac{1}{2}$.

This is the end of this lecture!

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